Published by Institute of Physics Publishing for SISSA/ISAS

Received: *June 15, 2004*
Accepted: *July 29, 2004*

Squashed giants: bound states of giant gravitons

Sergey Prokushkin and Mohammad M. Sheikh-Jabbari

Department of Physics, Stanford University
Stanford, CA 94305, U.S.A.
E-mail: prok@itp.stanford.edu, jabbari@itp.stanford.edu

Abstract: We consider giant gravitons in the maximally supersymmetric type-IIB plane-wave, in the presence of a constant NSNS B-field background. We show that in response to the background B-field the giant graviton would take the shape of a deformed three-sphere, the size and shape of which depend on the B-field, and that the giant becomes classically unstable once the B-field is larger than a critical value B_{cr}. In particular, for the B-field which is (anti-)self-dual under the SO(4) isometry of the original giant S^3, the closed string metric is that of a round S^3, while the open string metric is a squashed three-sphere. The squashed giant can be interpreted as a bound state of a spherical three-brane and circular D-strings. We work out the spectrum of geometric fluctuations of the squashed giant and study its stability. We also comment on the gauge theory which lives on the brane (which is generically a noncommutative theory) and a possible dual gauge theory description of the deformed giant.

Keywords: D-branes, Penrose limit and pp-wave background, AdS-CFT and dS-CFT Correspondence.

Contents

1. Introduction

Usual Dp-branes are $p+1$ dimensional objects which carry one unit of RR $p+1$ form charge and have tension $(l_s^{p+1} g_s)^{-1}$. One might try to construct p-branes whose worldvolume is (topologically) $R \times S^p$. Evidently such branes, unlike flat Dp-branes, cannot carry a net $p+1$ form RR charge. Moreover, in the absence of any other force acting on such branes, they would immediately collapse under their tension. Although a spherical brane cannot carry a net RR charge, it has (electric) dipole moment of such charges. In this respect such branes behave similarly to the usual fundamental strings and in particular supergravity states, hence these spherical branes were called *giant gravitons* [1]. One can use this dipole moment to exert a force on the brane which cancels off the tension force and stabilize the brane at a finite size. This can be done if the giant graviton is moving in a background with a non-zero (magnetic) flux of the corresponding $p+1$ form, and the size of the giant R_0 is related to the (angular) momentum J as $R_0^{p-1} \propto J$ [1]. So, to stabilize the giant we need two basic ingredients: background $p+1$ form flux and a moving brane. The simplest and most famous examples of such backgrounds are $AdS_p \times S^q$ geometries with $(p,q) = (5,5)$, $(4,7)$ or $(7,4)$. Various aspects of the giants in these backgrounds have been studied in [1]–[6].

Besides the AdS backgrounds, recently the plane-wave backgrounds have also been under intense study, for a review see [7, 8]. Plane-waves, as solutions of supergravity generically have a non-vanishing form flux and these fluxes can be used to stabilize spherical branes. The plane-wave background that we would focus on here is the maximally supersymmetric type-IIB background (we follow the notations and conventions of [8]):

$$ds^2 = -2dX^+dX^- - \mu^2(X^iX^i + X^aX^a)(dX^+)^2 + dX^idX^i + dX^adX^a\,, \tag{1.1a}$$

$$F_{+ijkl} = \frac{4}{g_s}\mu\,\epsilon_{ijkl}\,, \qquad F_{+abcd} = \frac{4}{g_s}\mu\,\epsilon_{abcd}\,, \tag{1.1b}$$

where $i, a = 1, 2, 3, 4$. This background has a globally defined light-like Killing vector $\partial/\partial X^-$. As discussed in [9] this background admits a stable three sphere giant graviton solution with the worldvolume along X^+ direction (the light-cone time) and three sphere is embedded in either the X^i or X^a directions. In particular note that X^- is transverse to the giant.

String theory σ-model on the plane-wave background is shown to be solvable in the light-cone gauge [8, 10]. In the light-cone gauge $X^+ = \tau$ (where τ is the worldsheet time) and X^- is a non-dynamical variable, completely determined through the transverse X^i and X^a directions, in particular [8]

$$\partial_\sigma X^- = \partial_\sigma X^i \partial_\tau X^i + \partial_\sigma X^a \partial_\tau X^a\,, \tag{1.2}$$

where σ, τ parametrize the worldsheet. One might then wonder whether giant gravitons, similarly to ordinary D-branes, have a perturbative description in terms of open strings ending on them with Dirichlet boundary conditions along the directions transverse to the brane. Noting (1.2) one would readily see that, independently of the boundary conditions on X^i or X^a directions, X^- would satisfy Neumann boundary condition. This implies that X^- should be along the brane, while in our giant graviton X^- is transverse to the brane. Therefore, giant graviton does not have a simple open string description and lots of the properties of the usual D-branes, e.g. the fact that the low energy effective field theory on a single three sphere giant is a U(1) supersymmetric gauge theory on $R \times S^3$ and that for K coincident giants the gauge symmetry enhances to U(K), if correct, are harder to realize or argue for. Although there is no simple open string picture for giants, studying the theory residing on the giant graviton, it was shown that there are (spike-type) BPS solutions on the giant three sphere which have the same physical behaviour one expects form the open strings ending on a spherical D-brane [9]; i.e. giant gravitons are spherical D-branes.

JHEP07(2004)077

Here we study behaviour of a giant in the plane-wave (1.1) when a constant NSNS B-field is turned on in the background. Turning on a constant NSNS background field would not change the geometry, as in the supergravity equations of motion only $H = dB$ appears which is vanishing in our case. Note that, existence of the selfdual five form flux in the background does not change this result. However, presence of the background five form flux affects dynamics of NSNS and RR two form fields so that different polarizations of the two form fields have different light-cone masses [8, 10]. One of the main motivations for studying this problem is that it may help with solving the long-standing problem of quantizing a p-brane ($p > 1$). Giant gravitons are particularly nice laboratories for attacking this problem mainly because, unlike the usual Dp-branes, their worldvolume is naturally compact and they have a finite volume and also the spectrum of fluctuations of the giant is discrete and (in the free field theory limit) is given by equally spaced integers [9].

For a flat Dp-brane, where a simple perturbative open string description is available, turning on the B-field along the brane simply amounts to replacing the Neumann boundary conditions with a mixed (Neumann and Dirichlet) boundary condition along the directions of the B-field [11]. As a result a Dp-brane in the B-field background behaves as a bound state of Dp and lower dimensional branes [11] and the low energy effective field theory on the brane is now a $p+1$ dimensional noncommutative gauge theory, e.g. see [12]. For the giant graviton case, however, we do not have the simple open string description and hence our analysis is more limited to the Born-Infeld action and supersymmetry algebra.

As mentioned earlier, spherical shape of the giant is a result of the balance between tension and forces coming from the form fluxes. As we will show in section 2, in the presence of the background B-field this balance is lost and hence the giant needs to reshape itself to adjust to the presence of B-field so that the shape of the giant is now a deformed three-sphere, the "squashed giant". This reshaping, which of course has no counterpart in the flat brane case, among the other things, would provide us with a chance of quantizing a submanifold of S^3 worldvolume, explicitly an $S^2 \in S^3$. This point will be addressed in some detail in section 2.5, where we show that it leads to the novel feature of quantization of the B-field. As we will show, if the B-field is larger than some critical value B_{cr} the giant cannot adjust itself to the B-field and becomes unstable. In other words, the squashed giant state only exists for $B < B_{cr}$. Once we calculated the shape of the deformed giant we look for a physical interpretation for the deformed or "squashed" giant. We will argue in section 2.6, that the squashed giant is indeed a bound state of a giant spherical D3-brane and circular D-strings wrapping on the S^3.

After establishing what the squashed giant is in section 2, in sections 3 and 4 we address the question of its stability. In section 3, we analyze small fluctuations of the giant and study corrections to the spectrum due to the deformation of the shape and presence of the B-field. In section 4, we show that squashed giant is a 1/4 BPS object. We use the supersymmetry analysis to argue that squashed giant is classically stable, however, quantum mechanically, through instanton effects, it would decay into the zero size branes and usual supergravity modes. We close the paper by a summary of our results, a proposal for a possible description of squashed giants in the dual $\mathcal{N} = 4$ gauge theory as well as a list of interesting questions which we did not address in this work.

2. Giant gravitons in a constant B-field background

In this section we first present the light-cone hamiltonian of a 3-brane in the plane-wave (1.1) with a constant *magnetic* background B-field, i.e. the B-field has no legs along the X^+, and X^- directions. Without loss of generality such B-field can be chosen to be along X^i directions. In other words we choose the only non-zero components of the B-field to be B_{ij} components. In section 2.2, we show that the round three sphere solution is no longer minimizing the potential, and in response to the background B-field the giant graviton takes a new shape. Giant gravitons in the non-constant background B-field and non-spherical giants, though in a different context, have been considered in [13, 14]. In section 2.3, we analyze the potential in some detail and compute the new shape of giant graviton which minimizes the potential. In section 2.5, we consider a particularly interesting example of a B-field background which preserves $SU(2) \times U(1)$ subgroup of $SO(4)$ isometry of the three-sphere (or the background plane-wave) and finally in section 2.6, we discuss that the "squashed" giant is indeed a bound state of a three sphere giant and circular D-strings.

2.1 Light-cone hamiltonian of giants in the plane-wave

To study a 3-brane in the plane-wave background we start with the Born-Infeld action:

$$S = \frac{1}{l_s^4 g_s} \int d\tau d^3\sigma \sqrt{-det\left(g_{\hat\mu\hat\nu} + \mathcal{F}_{\hat\mu\hat\nu}\right)} + \int C_4 + \int C_2 \wedge \mathcal{F} + \int \frac{1}{2}\chi \mathcal{F} \wedge \mathcal{F}\,, \tag{2.1}$$

where $\hat\mu, \hat\nu = 0,1,2,3$ indices correspond to the worldvolume coordinates τ, σ^r, $r = 1,2,3$. $g_{\hat\mu\hat\nu}$ is the induced metric on the brane:

$$g_{\hat\mu\hat\nu} = G_{\mu\nu}\partial_{\hat\mu}X^\mu \partial_{\hat\nu}X^\nu$$

where $G_{\mu\nu}$ is the background plane-wave metric and $X^\mu = (X^+, X^-, X^i, X^a)$ are the embedding coordinates of the three-brane. C_4, C_2 and χ are the background RR form fluxes. In our case we only have a non-zero C_4, whose field strength is the background five form flux of (1.1b), explicitly

$$C_{+ijk} = -\frac{\mu}{g_s}\epsilon_{ijkl}X^l\,, \qquad C_{+abc} = -\frac{\mu}{g_s}\epsilon_{abcd}X^d\,. \tag{2.2}$$

$\mathcal{F}_{\hat\mu\hat\nu} = b_{\hat\mu\hat\nu} + F_{\hat\mu\hat\nu}$ is the invariant $U(1)$ field of the brane, $F = dA$, A is the $U(1)$ gauge field on the brane and b is the pullback of the background NSNS two form field

$$b_{\hat\mu\hat\nu} = B_{\mu\nu}\partial_{\hat\mu}X^\mu \partial_{\hat\nu}X^\nu\,.$$

In the plane-wave background, due to the existence of a globally defined null Killing vector field [8], it is particularly useful to fix the light-cone gauge by taking[1]

$$X^+ = \tau\,, \qquad g_{\tau\sigma_r} = 0\,.$$

[1]Here we do not repeat details of the light-cone gauge fixing, for which the reader is referred to [9].

Following the analysis of [9] we obtain the light-cone hamiltonian density in the presence of a non-zero NSNS B-field:

$$H_{l.c.} = \frac{1}{2p^+} P^I P^I + V(X^i, X^a)\,, \tag{2.3}$$

with

$$\begin{aligned} V(X^i, X^a) &= \frac{\mu^2 p^+}{2}(X_i^2 + X_a^2) + \frac{1}{2p^+ g_s^2}\det(g_{rs} + b_{rs}) - \\ &- \frac{\mu}{6g_s}\Big(\epsilon^{ijkl} X^i \{X^j, X^k, X^l\} + \epsilon^{abcd} X^a \{X^b, X^c, X^d\}\Big)\,, \end{aligned} \tag{2.4}$$

where g_{rs} and b_{rs} are the spatial parts of pullbacks of the background metric and B-field onto the spatial part of the brane worldvolume. For our choice of the B-field

$$g_{rs} = \partial_r X^i \partial_s X^i + \partial_r X^a \partial_s X^a\,, \quad b_{rs} = \partial_r X^i \partial_s X^j\, B_{ij}\,, \qquad r, s = 1, 2, 3\,, \tag{2.5}$$

and

$$\{F, G, K\} = \epsilon^{prs} \partial_p F \partial_r G \partial_s K\,, \tag{2.6}$$

is the Nambu bracket, where the antisymmetrization is with respect to the worldvolume coordinates σ^r. Using the definition of determinant, the Nambu bracket (2.6) and the fact that B_{ij} is antisymmetric under exchange of i and j one can show that

$$\begin{aligned} \det(g_{rs} + b_{rs}) &= \frac{1}{6}\{X^i, X^j, X^k\}\{X^i, X^j, X^k\} + \frac{1}{6}\{X^a, X^b, X^c\}\{X^a, X^b, X^c\} + \\ &+ \frac{1}{2}\{X^i, X^j, X^a\}\{X^i, X^j, X^a\} + \frac{1}{2}\{X^i, X^a, X^b\}\{X^i, X^a, X^b\} + \\ &+ \frac{1}{2}\Big(\{X^i, X^k, X^m\}\{X^j, X^l, X^m\} + \{X^i, X^k, X^a\}\{X^j, X^l, X^a\}\Big) B_{ij} B_{kl}\,. \end{aligned} \tag{2.7}$$

The first two line of (2.7) is nothing but $\det g_{rs}$. As we see the whole hamiltonian can nicely be written in terms of the Nambu brackets.

In the case with $B = 0$, it was shown that [9] there are three minimum energy, half BPS solutions: one is point-like $X^i = X^a = 0$, and the other two are spherical three-branes, grown along the X^i directions and sitting at $X^a = 0$ or grown along the X^a directions and sitting at $X^i = 0$. The latter two solutions are related by the Z_2 symmetry which exchanges the X^i and X^a directions [9]. Here we focus on the $X^a = 0$ solutions. In the absence of the B-field and setting $X^a = 0$, the above potential is minimized at

$$\epsilon_{ijkl}\{X^j, X^k, X^l\} = 6 g_s \mu p^+ X^i\,. \tag{2.8}$$

The finite size solution of (2.8) can be expressed through

$$X^i = R_0 x^i\,, \qquad R_0^2 = \mu p^+ g_s\,, \tag{2.9}$$

where x^i's, which satisfy

$$x^i x^i = 1\,, \qquad \{x^j, x^k, x^l\} = \epsilon^{ijkl} x^i\,, \tag{2.10}$$

are the embedding coordinates of a unit three-sphere in $\mathbb{R}^4$.

2.2 Reshaping: response of the giant graviton to the B-field

Now we consider configurations with a non-zero, but constant (i.e. $dB = 0$) background B-field. Although a constant background B-field does not affect the (perturbative) dynamics of the closed strings, D-branes (giant gravitons) would feel the presence of the B-field which has both of its legs along the brane [12]. Using (2.7), it is straightforward to see that in the presence of the B_{ij} field, the three sphere giant grown in X^a directions (sitting at $X^i = 0$) is still a zero energy solution of the light-cone hamiltonian (2.3). Therefore, with the choice of the B_{ij} field, we focus on the giant which is grown in the X^i directions. Setting $X^a = 0$, the potential (2.4) can be rewritten as

$$\begin{aligned} V(X^i, X^a = 0) = & \frac{1}{2p^+}\left(\mu p^+ X^i - \frac{1}{6g_s}\epsilon_{ijkl}\{X^j, X^k, X^l\}\right)^2 + \\ & + \frac{1}{4p^+ g_s^2}\{X^i, X^k, X^m\}\{X^j, X^l, X^m\}\, B_{ij}B_{kl}\,. \end{aligned} \tag{2.11}$$

To find how a spherical three-brane responds to this B-field, let us consider fluctuations of the embedding coordinates around the spherical solution (2.9),

$$X^i = R_0\, x^i + Y^i\,. \tag{2.12}$$

Plugging this ansatz into (2.11) one gets the potential in the form of the expansion[2]

$$V = V^{(0)} + V_i^{(1)}\, Y^i + V_{ij}^{(2)}\, Y^iY^j + V_{ijk}^{(3)}\, Y^iY^jY^k + O[(Y^i)^4]\,. \tag{2.13}$$

The $V^{(0)}$ term, which is the zero point energy, is zero in the $B = 0$ case. This can be easily seen from the first line of (2.11), and is a result of the fact that the round three-sphere giant is a half SUSY state with zero light-cone energy [9]. In the $B \neq 0$ case,

$$V^{(0)} = \frac{1}{4p^+ g_s^2} R_0^6 \left(\frac{1}{2}B^2 + 2T_{ij}(B)x^ix^j\right)$$

where

$$B^2 = B_{ik}B_{ik}\,, \qquad T_{ij}(B) = B_{ik}B_{kj} + \frac{1}{4}\delta_{ij}B^2\,. \tag{2.14}$$

Note that T_{ij} is a symmetric traceless tensor (i.e. it lies in $\mathbf{9}$ of SO(4)). The potential (2.11) is a density, and to obtain contribution of $V^{(0)}$ to the total energy we need to integrate the potential over the unit three-sphere volume. Noting that[3]

$$\int_{s^3} d\Omega_3\, x^ix^j = \frac{1}{4}\delta_{ij} \tag{2.15}$$

[2]Note that the total potential energy of the brane is an integral of V over the spatial part of the brane worldvolume M_3:

$$E_p = \int_{M_3} d^3\sigma\, V(\sigma) = \int_{\mathbb{R}^4} d^4x\, \delta(x^ix^i - 1)\, V(x)\,.$$

We will use this fact and by-parts integration to obtain different expansion terms in (2.13) and (2.11).

[3]We have defined the measure of the integral so that $\int_{s^3} d\Omega_3 = 1$.

and choosing a constant B-field, i.e. $\{x^i, x^j, B_{kl}\} = 0$, we find

$$\begin{aligned} V^{(0)} &= \frac{1}{8p^+ g_s^2} R_0^6 B^2 = \mu \cdot \frac{1}{8}(\mu p^+)^2 g_s B^2 \\ &= \mu \cdot \frac{1}{8} g_2 B^2 = \mu \cdot \frac{1}{8 g_{\text{eff}}^2} B^2 \,, \end{aligned} \tag{2.16}$$

where g_2 is the effective coupling for strings in the plane-wave background (cf. appendix C) and $g_{\text{eff}}(= 1/\sqrt{g_2})$ is the effective coupling of the (gauge) theory residing on the giant graviton [9].

This result is somehow what one would expect: The constant background B-field can also be understood as a constant magnetic field on the brane (e.g. see [12]), and the energy, in units of μ, stored in a magnetic field B in a gauge theory with coupling g_{eff} is exactly the expression given in the second line of (2.16).

In the absence of the B-field the potential felt by the radial fluctuation, which is obtained by inserting $X^i = R_0 r x^i$ into (2.11), and setting $B = 0$:

$$V(r) = \frac{R_0^6}{2p^+ g_s^2} r^2 (r^2 - 1)^2 = \mu \cdot \frac{1}{2 g_{\text{eff}}^2} r^2 (r^2 - 1)^2 \tag{2.17}$$

has a maximum at $r^2 = 1/3$. (This is the potential studied in [1, 2, 3].) The value of the potential at this maximum is

$$V^0_{\max} = \mu \, \frac{2}{27 g_{\text{eff}}^2} \,. \tag{2.18}$$

One would then expect that when $V^{(0)}$ becomes equal to $V^0_{\max}$ or larger, the potential loses the minimum, so that the giant graviton becomes unstable and rolls down toward the minimum at $r = 0$. This would happen for B-fields larger than the critical value

$$B^{(0)}_{cr} = \frac{4}{3\sqrt{3}} \,. \tag{2.19}$$

The above estimate for B_{cr} is a rough one and there are two points which should be taken into account. Adding the B-term increases the energy at the maximum and also changes the value of r at which the potential is maximized. In fact there is a term proportional to $r^6 B^2$ which should be added to (2.17). This effect would increase B_{cr} from (2.19) level to about $\sqrt{4/3}$. This will be discussed in more detail in sections 2.3 and 2.5. Moreover, and as we will show momentarily, besides the resizing, shape of the giant can (will) change, lowering the "vacuum" energy from $V^{(0)}$ as well as increasing $V^0_{\max}$.

The next term in (2.11) is the term linear in Y^i. This term is responsible for the resizing and also reshaping of the giant graviton. As discussed in [9] and can also be readily seen from (2.11), in the absence of the B-field this term vanishes and hence, similarly to the $V^{(0)}$ term, this term is second order in B. It is straightforward to show that[4]

$$\begin{aligned} V_i^{(1)} Y^i &= \frac{3}{4p^+ g_s^2} \, B^2 R_0^5 \, x_i Y^i - \frac{R_0^5}{p^+ g_s^2} \, T_{ij}(B) x^i Y^j \\ &= \mu \, \frac{1}{4 g_{\text{eff}}} \left(3B^2 \delta_{ij} - 4T_{ij}\right) x_i (\sqrt{\mu p^+} Y_j) \,. \end{aligned} \tag{2.20}$$

[4]Note that the "canonically normalized" fluctuation is $\sqrt{\mu p^+} Y_i$, cf. discussions of section 2.4 of [9].

JHEP07(2004)077

The $V_i^{(1)} Y^i$ term of the potential (2.20) results in a force acting on the spherical brane which consists of two components. The component proportional to $x_i Y^i$ is responsible for an overall resizing of the brane, whereas the component proportional to $T_{ij}(B)\, x^i Y^j$ causes changes in its shape.

To find a new size and shape of the brane, we introduce an ansatz with a corrected background,

$$X^i = R_0\, x^i + Y_0^i(B) + Z^i\,, \tag{2.21}$$

where $Y_0^i(B)$ is to be fixed requiring that there is no linear term in Z^i in the expansion of the potential. Explicitly, Y_0^i should satisfy

$$V_i^{(1)} + 2V_{ij}^{(2)}\, Y_0^j + 3V_{ijk}^{(3)}\, Y_0^j Y_0^k + \cdots = 0\,. \tag{2.22}$$

Assuming that the reshaping and resizing are small, i.e. $Y_0^i \ll R_0$, we can neglect higher order terms in (2.22). Since the force term is proportional to the B-field, small Y_0^i assumption is equivalent to a similar assumption on the B-field, $B^2 \ll 1$, in which case (2.22) reduces to

$$V_i^{(1)} + 2V_{ij}^{(2)}\, Y_0^j = 0\,. \tag{2.23}$$

All the geometric fluctuations of a giant three-sphere have been analyzed in [9], and in particular it was noted that these fluctuations can be classified in terms of SO(4) harmonics or, in other words, by irreducible representations of SO(4). Moreover, it was noted that $V_{ij}^{(2)}$ is diagonalized by SO(4) spherical harmonics. Hence Y_0^i that solves (2.23) should have the same SO(4) harmonic structure as the force term $V_i^{(1)}$, explicitly

$$Y_0^i(B) = R_0\, \left[(r(B) - 1)\, x^i + \lambda S_{ij}(B)\, x^j\right], \tag{2.24}$$

where S_{ij} similarly to T_{ij} is a symmetric traceless tensor of SO(4), and r and λ are two variables which should be solved for (as functions of B). Solving (2.23) we find

$$\begin{aligned} &S_{ij} = T_{ij}\,, \\ &r(B) = 1 - \frac{3}{16}B^2 + \mathcal{O}(B^4)\,, \qquad \lambda(B) = \frac{1}{4} + \mathcal{O}(B^2)\,. \end{aligned} \tag{2.25}$$

As we see, the correction to the radius is negative. That is, the size of the giant graviton is reduced under the B-field, while its shape is "squashed" with the "stress" tensor of the B-field, $T_{ij}(B)$. It is worth noting that the new shape and size do not depend on $g_{\rm eff}$ and are only functions of the B field.

Before moving to a complete and general analysis of reshaping and resizing we would like to reconsider our zero point energy analysis. If the reshaped giant graviton is stable, along with our force arguments we expect that reshaping should decrease the zero point energy. The corrected zero point energy is then obtained as

$$\tilde{V}^{(0)} = V^{(0)} - V_{ij}^{(2)}\, Y_0^i Y_0^j + \cdots\,, \tag{2.26}$$

JHEP07(2004)077

which up to the fourth order in B is

$$\begin{aligned}\tilde{V}^{(0)} &= V^{(0)} - (V_0)^{(2)}_{ij}\, Y_0^i Y_0^j + \mathcal{O}(B^6)\\ &= \mu\cdot\frac{1}{8g_{\rm eff}^2}\left[B^2 - \frac{1}{16}\Big(9(B^2)^2 + 4T^2(B)\Big)\right] + \mathcal{O}(B^6)\\ &= \mu\cdot\frac{1}{8g_{\rm eff}^2}B^2\left(1-\frac{3}{16}B^2\right)^3 - \mu\cdot\frac{1}{32g_{\rm eff}^2}T^2 + \mathcal{O}(B^6)\,,\end{aligned} \tag{2.27}$$

where $T^2 = T_{ij}T^{ij}$. To perform the above computation we have used the formula [9]

$$(V_0)^{(2)}_{ij}\, Y_0^i Y_0^j = \frac{\mu^2 p^+}{2}(Y_0^i + \mathcal{L}_{ij}Y_0^j)^2\,, \qquad \mathcal{L}_{ij} = x_j\partial_i - x_i\partial_j\,, \tag{2.28}$$

and the expression for Y_0^i (2.24), (2.25). It is worth noting that in the third line of (2.27) the first term is the energy stored in the magnetic field B on a three-sphere of radius $R_0\, r$ (it encodes resizing of the giant graviton), whereas the second term, which is proportional to T^2, comes from reshaping of the brane. In other words (2.27) is the energy stored in a magnetic field B on the squashed giant.

2.3 Detailed analysis of the potential

In the previous section we assumed that the reshaping and resizing are small, which is equivalent to a similar assumption on the B-field, $B^2 \ll 1$. In the lowest order in the B-field, we found that reshaping of the brane is described by the lowest (linear in x^i) harmonics of SO(4). In this section, we study the potential for arbitrary values of the B-field, and show that the reshaping is always described by the lowest harmonics of SO(4). In other words, the ansatz (2.24) remains valid for the large B-field, and one does not have to include higher SO(4) harmonics.

To study the effects of a large B-field, it is convenient to use the equations of motion for the embedding $X^i = X^i(\sigma)$ that extremize the potential (2.11). To obtain these equations we compute variation of the potential δV under the variations of the embedding δX^i, and equate it to zero. The result is

$$\begin{aligned}&2R_0^4\, X^i - \frac{4}{3}\, R_0^2\, \epsilon_{ijkl}\{X^j, X^k, X^l\} + \frac{1}{6}\,\epsilon_{ijkl}\epsilon_{jmnp}\{\{X^m, X^n, X^p\}, X^k, X^l\}-\\ &\quad -2\{\{X^j, X^l, X^m\}, X^k, X^m\}\, B_{ij}B_{kl} - \{X^m, X^k, \{X^j, X^l, X^i\}\}\, B_{mj}B_{kl} = 0\,.\end{aligned} \tag{2.29}$$

Now let's plug the ansatz of the form

$$X^i(x) = \sum_n S^i_{i_1 i_2 \ldots i_n}(B)\, x^{i_1}x^{i_2}\cdots x^{i_n}\,, \tag{2.30}$$

into (2.29) and note that except the first term in (2.29) all the other terms, in particular the terms proportional to the B-field, only involve Nambu brackets of X^i's. Using the Leibnitz rule for the Nambu bracket and equation (2.10) one observes that the equations (2.29) reduce to a system of algebraic equations that relate the components $S^i_{i_1 i_2 \ldots i_n}$ of ranks n, $n' = \frac{n+2}{3}$, and $n'' = \frac{n+4}{5}$. This system does not close for any finite number of terms in (2.30), except for the trivial solution $X^i = 0$ and the linear solution with $n = 1$. These two solutions are in agreement with the results of the perturbative analysis of the previous section, where we found the point-like solution and the squashed giant graviton solution (2.21), (2.24), (2.25). Since for a small B we didn't see any solutions with infinitely many higher harmonics, which could in principle solve (2.29), we do not expect them to appear for arbitrary values of B. Therefore, the linear ansatz (2.24) already includes all the harmonics needed for arbitrary values of the B-field. Then, noting (A.3),

$$X^i = R_0\, D_{ij}(B)\, x^j\,, \qquad D_{ij}(B) = r(B)\delta_{ij} + \lambda(B) T_{ij}(B)\,, \tag{2.31}$$

is the most general ansatz which solves (2.29). (Indeed, using (A.3) it easy to see that all second rank tensors made out of higher powers of B_{ij} are proportional to either δ_{ij} or T_{ij}.) Thus, for a given B-field the problem reduces to the study of the potential as a function of just two variables r and λ.

Using the (anti-)selfdual decomposition (A.1) and formulas (A.4), (A.5), it is convenient to introduce the "balance parameter"

$$\gamma^2 = \frac{4T^2}{(B^2)^2} = \frac{4(B^+)^2(B^-)^2}{[(B^+)^2 + (B^-)^2]^2}\,, \tag{2.32}$$

in addition to $B^2 = B_{ij}B^{ij}$, to describe the B-field background. This parameter ranges from zero to one, vanishes for an (anti-)selfdual B-field, and is equal to 1 for a "balanced" ($B^+ = \pm B^-$) B-field. Also, it is convenient to use the "shape parameter"

$$s = \frac{1}{4}\gamma\lambda B^2 \tag{2.33}$$

instead of λ. The change of variables from λ to s would become ambiguous for $\gamma = 0$, the (anti-)selfdual B-field case, which would be studied separately in section 2.5. In this section we only consider $\gamma \neq 0$. Plugging the ansatz (2.31) into the potential (2.11) we find

$$\begin{aligned} V = \frac{R_0^6}{8p^+ g_s^2}\Bigg[&\mathrm{Tr}\, D^2 - 8\,\det D + \frac{1}{6}((\mathrm{Tr}\, D^2)^3 - 3\;\mathrm{Tr}\, D^2\;\mathrm{Tr}\, D^4 + 2\;\mathrm{Tr}\, D^6) + \\ &+ \mathrm{Tr}(D^2 B D^4 B) - \frac{1}{2}\,\mathrm{Tr}(D^2 B D^2 B)\;\mathrm{Tr}\, D^2\Bigg]. \end{aligned} \tag{2.34}$$

Making use of the formulas given in appendix A and introducing the parameters γ and s (2.32), (2.33), we obtain

$$\begin{aligned} V = \frac{\mu}{8g_{\rm eff}^2}\Big\{&4[r^2(r^2-1)^2 + s^2(s^2-1)^2 + r^2 s^2(4 - r^2 - s^2)] + \\ &+ B^2\,(r^2 - s^2)^2(r^2 + s^2 - 2\gamma\, rs)\Big\}. \end{aligned} \tag{2.35}$$

Note that all the dependence on $g_{\rm eff}$ has been factored out and hence the reshaping and resizing are independent of $g_{\rm eff}$ and are only functions of B-field. This potential has an obvious symmetry

$$r \leftrightarrow s\,, \tag{2.36}$$

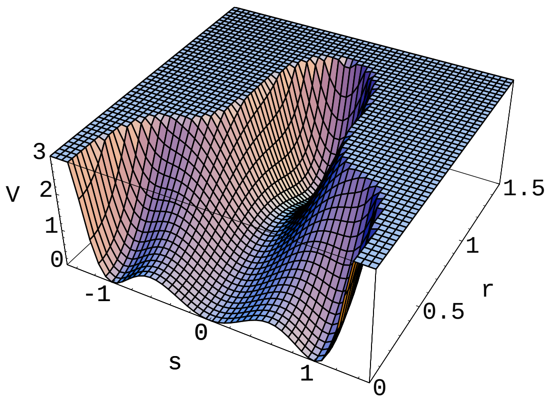

Figure 1: Potential as a function of r and s for $\gamma = 1$ and $B = 0$.

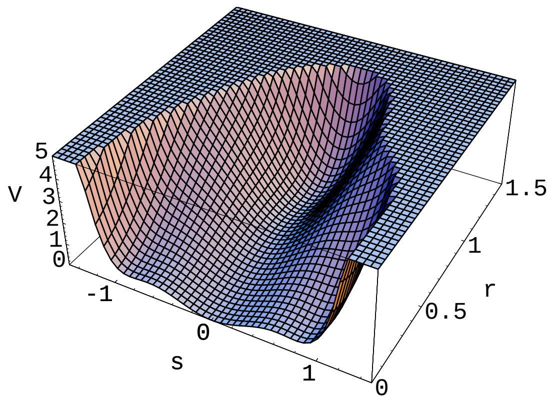

Figure 2: Potential as a function of r and s for $\gamma = 1$ and $B = 1$.

which in fact is a part of SO(4) rotational symmetry of the problem. To see this, notice that the matrix $\mathcal{O}_{ij} = \frac{2}{T} T_{ij}$, $T = \sqrt{T^2}$, is orthogonal, $\mathcal{O}^T\mathcal{O} = \mathbb{I}$. In other words,

$$X^i = \frac{2}{T}\, T_{ij} x^j \tag{2.37}$$

gives another embedding of a round S^3 into $\mathbb{R}^4$. Using the ansatz (2.31) rewritten in the form

$$X^i(r,s) = R_0(r\delta_{ij} + s\mathcal{O}_{ij})x^j\ , \tag{2.38}$$

we find

$$\mathcal{O}_{ij} X^j(r,s) = X^i(s,r)\,. \tag{2.39}$$

Since $\mathcal{O}_{ij}$ is a constant SO(4) rotation matrix,

$$V(X^i, X^a) = V(\mathcal{O}_{ij} X^j, X^a)\,.$$

Profile of the potential (2.35) for the case of the "balanced" B-field ($\gamma = 1$) for different values of $B = \sqrt{B^2}$ is shown in figures 1–4. As we see in figure 1, for the vanishing B-field we have minima at $s = 0$ and $r = 0$, $r = 1$. These are the point-like and spherical brane solutions studied in [1]. Note that the minima at $r = 0$, $s = \pm 1$ describe the same spherical brane as the minimum at $s = 0$, $r = 1$, due to the symmetry (2.36) and additional symmetry $r \leftrightarrow -s$ that appears at $B = 0$. The plot in figure 2 corresponds to $B = 1$ and shows the situation when the minimum at $s = 0$, $r \lesssim 1$ is lifted and is about to disappear, and the brane is about to roll toward $X = 0$ vacuum. This can be easily seen on the $r = 0$ section of the plot, given the symmetry (2.36). Using numerical analysis we found that the minimum disappears at $B_{cr} \approx 1.184$. Figure 3, corresponding to $B = 1.5$, shows the situation when the minimum at $r \neq 0$ has already disappeared, so that the only minimum left is point-like $r = s = 0$. One observes here how large B effects related to the second term in (2.35) begin to dominate. The potential starts developing two valleys at $r = \pm s$, which become dominant in the $B = 10$ case shown in figure 4.

It is also important to know how the shape of the brane depends on the value of B-field, before the brane shrinks to a point. The shape of the brane is described by the "shape parameter" s (2.33), which is equal to zero for a round sphere solution. It turns

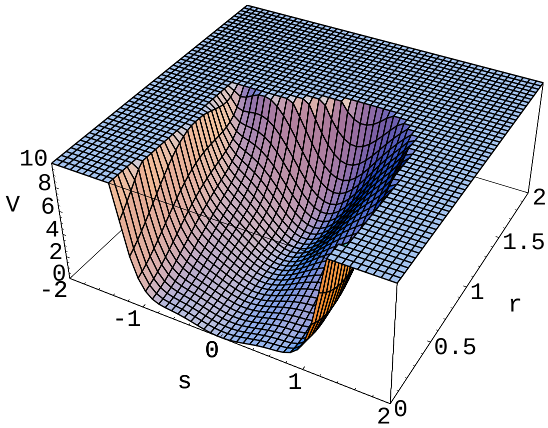

Figure 3: Potential as a function of r and s for $\gamma = 1$ and $B = 1.5$.

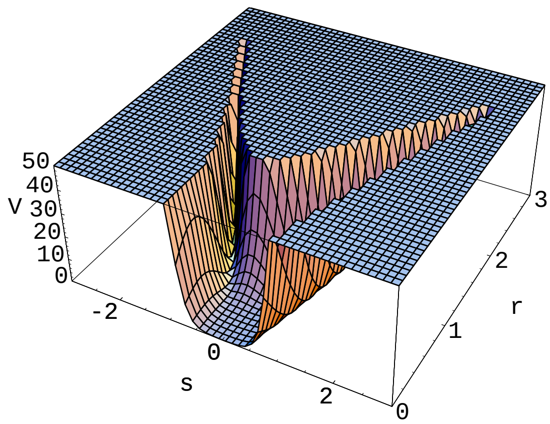

Figure 4: Potential as a function of r and s for $\gamma = 1$ and $B = 10$.

out, however, that it is more convenient to introduce the parameter

$$q_c = \frac{r+s}{r-s}\,, \tag{2.40}$$

which is equal to one for a round sphere. (The subscript c on q shows that this parameter measures the out-of-sphericity for the closed string metric.) The dependence of q_c on B, for $\gamma = 1$ B-field is depicted in figure 5. One observes that $q_c(B)$ grows with B monotonically almost everywhere, except the region of B close to the critical value $B_{cr} \approx 1.184$, where q_c decreases with B.

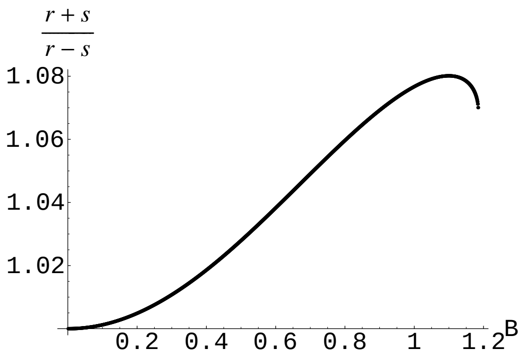

Figure 5: Parameter $q_c = \frac{r+s}{r-s}$ as a function of B for $\gamma = 1$. At $B \approx 1.184$ the brane shrinks to a point, and the parameter q_c can no longer be used to describe the shape of the brane.

For the rank one B-field ($\gamma = 1$) in the large B limit,

$$B \to \infty\,, \qquad \frac{B^2}{g_{\rm eff}^2} = \text{const.}\,, \tag{2.41}$$

the first term in (2.35) disappears, and the valleys $r = \pm s$ become flat directions. In other words, it costs no energy to roll in the directions $r = \pm s$. To find the shape of the brane in this case we note that the embedding (2.38) takes the form

$$X^i(r,s) = R_0 r(\delta_{ij} \pm \mathcal{O}_{ij})x^j\,. \tag{2.42}$$

In the next section we show that $\mathcal{O}_{ij}$ can be brought into a diagonal form (2.47). Taking "+" or "−" sign in (2.42), one finds the embeddings

$$\begin{aligned} X^{1,2} &= R_0 r x^{1,2}\,, \qquad & X^{3,4} &= 0\,, \\ X^{3,4} &= R_0 r x^{3,4}\,, \qquad & X^{1,2} &= 0\,, \end{aligned} \tag{2.43}$$

where now r can take any arbitrary value and hence (2.43) describes a plane which grows in the $X^{1,2}$ or $X^{3,4}$ directions.

Given the definition of g_{eff} (2.16), the limit (2.41) is equivalent to

$$B \to \infty\,, \qquad g_2 B^2 = \text{const.} \tag{2.44}$$

where g_2 is the coupling for the closed strings in the plane-wave background. In other words, in this limit B-field is large and the string coupling is small. In this regime the brane loses one of its dimensions and becomes a two dimensional plane.

2.4 Open string parameters

Equation (2.38) (or (2.31)) is basically determining shape of the three-brane giant embedded in $\mathbb{R}^4$. Using these it is straightforward to compute the induced metric on the brane

$$\begin{aligned} ds^2 &= \frac{\partial X^k}{\partial x^i}\frac{\partial X^k}{\partial x^j} dx^i dx^j \\ &= R_0^2\left[(r^2+s^2)\delta_{ij} + 2rs\mathcal{O}_{ij}\right] dx^i dx^j\,, \end{aligned} \tag{2.45}$$

where r and s, which are functions of B, are minimizing the potential (2.35). The metric (2.45) is in fact giving the shape of the brane viewed by closed strings, i.e. (2.45) is the closed string metric.

Due to the term proportional to $\mathcal{O}_{ij}$ for a generic B-field SO(4) rotational isometry of the giant is reduced to a $\mathrm{U}(1)\times\mathrm{U}(1)$ subgroup. To see this note that we can always find a basis in which $B_{12} = -B_{21}$ and $B_{34} = -B_{43}$ are the only non-zero components of the B-field. In this basis the two U(1) are simply rotations in 12 and 34 planes and the non-vanishing components of T_{ij} are

$$T_{11} = T_{22} = \frac{1}{2}(B_{12}^2 - B_{34}^2)\,, \qquad T_{33} = T_{44} = -T_{11} \tag{2.46}$$

and $T^2 \equiv T_{ij}T^{ij} = 4T_{11}^2$. Without loss of generality we take $T_{11} > 0$. Then, for the $T \neq 0$ case, i.e. when $B \neq 0$ or $B_{12} \neq \pm B_{34}$,

$$\mathcal{O}_{ij} = \mathrm{diag}(1,1,-1,-1)\,. \tag{2.47}$$

(The case $B_{12} = \pm B_{34}$ which correspond to (anti-)selfdual B-field will be considered separately in section 2.5.)

To work out the explicit form of the metric (2.45), we adopt the coordinate system

$$\begin{aligned} x^1 + ix^2 \equiv z_1 &= \cos\frac{\theta}{2}\, e^{i\alpha}\,, \qquad 0 \le \alpha,\beta \le 2\pi\,, \\ x^3 + ix^4 \equiv z_2 &= \sin\frac{\theta}{2}\, e^{i\beta}\,, \qquad 0 \le \theta \le \pi\,, \end{aligned} \tag{2.48}$$

in which the closed string metric takes the form

$$\begin{aligned} ds^2 &= R_0^2\left[(r+s)^2 dz_1 d\bar{z}_1 + (r-s)^2 dz_2 d\bar{z}_2\right] \qquad (2.49) \\ &= \frac{R_0^2}{4}\left[(r^2+s^2-2rs\cos\theta)\; d\theta^2 + 4(r+s)^2\sin^2\frac{\theta}{2}\, d\alpha^2 + 4(r-s)^2\cos^2\frac{\theta}{2}\, d\beta^2\right] \end{aligned}$$

The embedding (2.48) explicitly demonstrates the two U(1) symmetries (as rotations in z_1, z_2 planes) and also the $r \leftrightarrow s$ symmetry. There is another Z_2 symmetry which exchanges z_1 and z_2 (together with $s \leftrightarrow -s$). These Z_2 symmetries are reminiscent of the original SO(4). From the metric (2.49) it becomes clear that q_c defined in (2.40) is indeed a measure of deformation of the round sphere.

By now it is well-known that D-branes in the constant B-field background, probed by the open strings, behave as noncommutative surfaces [12, 15], e.g. the low energy effective theory residing on these branes is a noncommutative SYM theory [16]. Moreover, the metric and the coupling viewed by open strings is different than those of closed strings. In our case this implies that the shape of the giant graviton seen by open strings is different than the one given through metric (2.49). Note that in our case, unlike the flat D-brane case, the Seiberg-Witten limit [12] does not lead to the decoupling of bulk closed strings.[5] Nevertheless, the notion of open string parameters is a useful one.

In [12], a prescription of calculating the open string metric G_{rs} and the noncommutativity parameter Θ^{rs} in terms of closed string ones was introduced:[6]

$$G^{rs} = \frac{1}{2}\left[(g_c+b)^{-1} + (g_c-b)^{-1}\right] \tag{2.50a}$$

$$\Theta^{rs} = \frac{1}{2}\left[(g_c+b)^{-1} - (g_c-b)^{-1}\right], \tag{2.50b}$$

where g_c is the induced closed string metric (2.49), and

$$\begin{aligned} b &= B_{ij}\frac{\partial X^i}{\partial x^k}\frac{\partial X^j}{\partial x^l}dx^k \wedge dx^l \\ &= R_0^2\left[B_{12}(r+s)^2 i dz_1 \wedge d\bar{z}_1 + B_{34}(r-s)^2 i dz_2 \wedge d\bar{z}_2\right] \\ &= \frac{R_0^2}{4}\sin\theta\left[B_{12}(r+s)^2 d\theta \wedge d\alpha + B_{34}(r-s)^2 d\theta\wedge d\beta\right]. \end{aligned} \tag{2.51}$$

Using the above formulas it is straightforward to work out open string parameters for the general rank two B-field. However, here we only present the explicit expressions for a rank

[5] For the flat D-branes in the B-field background, it is possible to take $\alpha' \to 0$ limit in such a way that the open string metric (and hence the open string mass scale) and noncommutativity parameter Θ are held fixed while the closed strings of the bulk become very massive [12]. Massless (supergravity) modes of the closed strings are also decoupled because the closed string coupling is also sent to zero, while the open string coupling is kept finite. In the plane-wave case, however, all the closed string modes, including the supergravity modes are massive (i.e. they have a non-zero light-cone mass) [8]. Moreover, all the physical modes of the giant three sphere which correspond to geometric fluctuations of the giant, are massive [9]. The mass scale for both of the closed strings and the giant fluctuations (the open string modes) is μ. Turning on the B-field, as we will study in section 3, in the range that we can still use the giant graviton description ($B < B_{cr}$) would not change the spectrum of the fluctuations of the giant very much. So, for the squashed giant we do not have a decoupling limit similarly to the Seiberg-Witten case.

[6] As was discussed in [12, 17] open string parameters are not invariant under the U(1) gauge transformation which rotates B-field into the U(1) gauge field on the brane, F. In particular, the above prescription is in a gauge in which the background (magnetic) field on the brane, F, is set to zero.

one B-field, $B_{12} = 0$, $B_{34} = B/\sqrt{2}$:[7]

$$ds^2_{\rm close} = \frac{R_0^2}{4}\left[G_{\theta\theta}\left(d\theta^2 + 4\frac{(r-s)^2\cos^2\frac{\theta}{2}}{r^2+s^2-2rs\cos\theta}d\beta^2\right) + 4(r+s)^2\sin^2\frac{\theta}{2}d\alpha^2\right] \tag{2.52a}$$

$$\Theta_{\theta\beta} = -\frac{BR_0^2}{4\sqrt{2}}(r-s)^2\sin\theta\left(1+\frac{B^2/2}{\frac{(r+s)^2}{(r-s)^2}+\cot^2\frac{\theta}{2}}\right), \tag{2.52b}$$

where $G_{\theta\theta} = r^2 + s^2 - 2rs\cos\theta + \frac{1}{2}(r-s)^2\sin^2\frac{\theta}{2}B^2$. Note that indices on Θ are lowered and raised by the open string metric.

2.5 (Anti-)self-dual B-field background

For a generic B_{ij}, T_{ij} is non-zero and hence, generically the shape of the giant graviton is deformed. However, if B-field is self-dual or anti-selfdual (i.e. B is in $(\mathbf{3},\mathbf{1})$ or $(\mathbf{1},\mathbf{3})$ of $\mathrm{SU}(2)\times\mathrm{SU}(2)\simeq\mathrm{SO}(4)$) T_{ij} is identically zero, and shape of the giant, viewed by closed strings, remains a round S^3 (while we still have resizing). This can directly be seen from the force term (2.20). In this section we consider this particular B-field.

This case can be analyzed by setting $s = 0$ in the potential (2.35):

$$V_{ASD} = \mu\frac{1}{2g^2_{\rm eff}}\left[r^2(r^2-1)^2 + \frac{B^2}{4}r^6\right]. \tag{2.53}$$

For all values of B this potential has a minimum at $r = 0$. For values of B-field less than the critical value

$$B_{cr} = \sqrt{4/3}\,, \tag{2.54}$$

this potential has another minimum at

$$r^2_{\rm min} = \frac{2}{12+3B^2}(4+\sqrt{4-3B^2}) = \frac{1}{2-\sqrt{1-\frac{3}{4}B^2}} \tag{2.55}$$

as well as a maximum. The value of the potential at this minimum is

$$V_{\rm min} = \mu\frac{B^2}{4g^2_{\rm eff}}\cdot\left(\frac{1}{2-\sqrt{1-\frac{3}{4}B^2}}\right)^2\cdot\frac{1}{1+\sqrt{1-\frac{3}{4}B^2}}\,. \tag{2.56}$$

This potential has been depicted in figure 6 for values of the B-field below and above B_{cr}. Note that the critical value of B-field where the giant graviton becomes classically unstable depends on the "balance parameter" γ. As we discussed earlier, for $\gamma = 1$ (rank one B-field) $B_{cr} \approx 1.184$ which is slightly more than in the selfdual case, $\sqrt{4/3}$. This can be understood simply by noting that the reshaping, on top of the resizing, would also decrease the energy (this can be seen from e.g. (2.27)).

[7] Note that rank one B-field corresponds to the balance parameter $\gamma = 1$. This can be readily seen from (2.46).

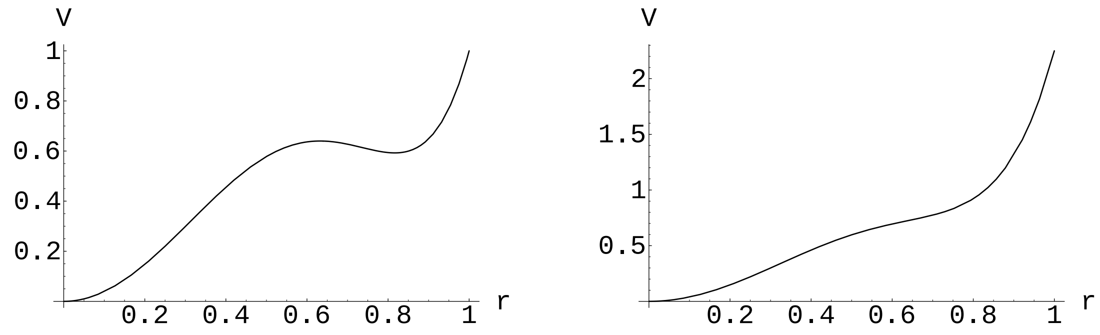

Figure 6: Potential as a function of r ($\gamma = 0$, $s = 0$) for $B = 1$ (left) and $B = 1.5$ (right). The critical value of B-field is $B_{cr} = \sqrt{4/3} \approx 1.155$.

In the (anti)-selfdual case, although shape of the giant remains SO(4) invariant, due to the background B-field this symmetry is reduced to a SU(2) $\times$ U(1). To see this, let us start with an anti-selfdual B-field, $B_{12} = -B_{34} = B/2$. In the "polar coordinates" adopted in eq. (B.1) of appendix B the pullback of this B-field on the round sphere of radius $R \equiv R_0 r_{\rm min}$ is

$$b = \frac{BR^2}{8} \sin\theta d\theta \wedge d\phi. \tag{2.57}$$

As we see, b is invariant under the U(1) acting on ψ coordinate, and the SU(2) acting on the two sphere parameterized by θ, ϕ (cf. appendix B).

This SU(2) $\times$ U(1) symmetry can also be explicitly seen in the open string parameters, and in particular open string metric. Working out the open string parameters using formulas of previous section we have

$$ds^2_{\rm close} = \frac{R^2}{4}\left(1 + \frac{B^2}{4}\right)\left[(d\theta^2 + \sin^2\theta d\phi^2) + \frac{1}{1 + B^2/4}(d\psi + \cos\theta d\phi)^2\right] \tag{2.58a}$$

$$\Theta_{\theta\phi} = -\frac{BR^2}{8}\left(1 + \frac{B^2}{4}\right)\sin\theta\,. \tag{2.58b}$$

The open string metric is a squashed sphere S^3_q with the squashing parameter (cf. appendix B)

$$q^2 = \frac{1}{1 + B^2/4}\,. \tag{2.59}$$

Note that $q \leq 1$. The noncommutativity parameter Θ, as we expect, is a constant two form on the S^2 base (it is proportional to the volume form of the base). It is worth noting that Θ is the flux of the magnetic field $B/2$ through the S^2 base of radius $R\sqrt{1 + B^2/4}/2$.

Upon quantization (of open strings) the S^2 base with the noncommutativity $\Theta_{\theta\phi} \propto \sin\theta$ becomes a fuzzy two sphere, S^2_F e.g. see [18]. Consider the following embedding in a three dimensional noncommutative space with coordinates X^r, $r = 1, 2, 3$:

$$[X^r, X^s] = il\epsilon^{rsp}X_p\,, \qquad X_r^2 = R^2\,. \tag{2.60}$$

The fuzzy two sphere is described by a finite dimensional representation of the SU(2) algebra whose generators are X^r/l; the radius of the sphere and the size of the matrices N

are related as [19]

$$\left(\frac{R}{l}\right)^2 = \frac{1}{4}(N^2 - 1)\,. \tag{2.61}$$

If we use the usual polar coordinates and write X^r as functions of θ and ϕ, (2.60) implies that

$$[\theta, \phi] = i\Theta_\theta^\phi \sim i\frac{1}{\sin\theta}\left(\frac{l}{R}\right).$$

Comparing the above with (2.58b) we learn that the B-field should be quantized. For small B-field (or large N), i.e.

$$B = \frac{4}{N}\,. \tag{2.62}$$

2.6 Reshaped giant as a giant bound state

In the case of a flat D-brane, it was shown that [11] a Dp-brane in a constant background rank one magnetic B-field which has both legs along the brane behaves as a (non-marginal) bound state of p- and $(p-2)$- branes and the background B-field is giving the density of the distribution of the RR charge corresponding to $(p-2)$-brane on the Dp-brane worldvolume. If the worldvolume of Dp-brane is along $012\ldots p$ directions and the B-field is along $B_{p-1,p}$ then the corresponding D$(p-2)$-branes are along $012\ldots p-2$ directions. From the string theory point of view, formation of this bound state can be understood as follows. Although both of Dp and D$(p-2)$-branes are individually half BPS configurations, a system consisting of both of them is not. This can be seen from the fact that the open strings stretched between the a D$(p-2)$- and Dp- brane would become tachyonic once the separation between branes is of order of string scale (or smaller). Due to the attractive force between the two branes formation of this tachyon is inevitable and finally the two branes would become coincident, at which point the tachyon is condensed and we end up with a (half) BPS brane, the Dp-D$(p-2)$ brane bound state [11]. The process of "dissolving" of a $(p-2)$-brane into a p-brane, from the $p+1$ dimensional gauge theory of p-brane viewpoint, is equivalent to turning on one unit of the magnetic flux.

In the spherical brane and giant graviton case the story is somewhat different. Let us focus on the three-sphere giants in the plane-wave background. Since there is no RR three form flux in the background, a circular D-string is not stable and the arguments made in the flat D-brane case [11] should be modified for this case. Moreover, spherical threebranes and circular D-strings are not carrying a net RR charge, however, they are (electric) dipole moments of the corresponding RR field.

Although the net RR charge of the giants is zero, locally they behave like a usual D-brane and hence we expect the system of a three sphere giant in the background B-field to (locally) behave like a bound state of a threebrane and D-strings. The simplest configuration of these giant bound states is coming from the self-dual or (anti)-selfdual B-field case we studied in the previous section. If the pullback of the B-field along the giant three sphere is along a S^2 parametrized with θ, ϕ then the *dissolved* circular D-strings should be along $\sigma = d\psi \pm \cos\theta d\phi$ direction (the $\pm$ sign corresponds to the self-dual or anti-self-dual B-field). The density of the D-string RR dipole moment is then proportional to $\Theta^{\theta\phi}$.

JHEP07(2004)077

Dissolving of circular string giants into the three sphere giant can be seen from the open string metric (2.58): Due to the tension of the circular D-strings the size of the S^1 fiber along which they are wrapping is smaller than the S^2 base. Furthermore note that the overall size of the giant due to this extra tension of the D-string giants has been reduced (this e.g. can be seen from figure 6).

3. Spectrum of fluctuations of the squashed giant

Knowing the correct shape of the brane, one can find the spectrum of small fluctuations around this configuration. In [9], the spectrum of small fluctuations around the spherically shaped branes was computed. It includes both massive and massless (or zero) modes. The zero modes are not physical modes and are gauge degrees of freedom, reminiscent of the area preserving diffeomorphisms on the three sphere giant, while the non-zero modes, which are of course all massive, correspond to the geometric fluctuations of the giant in the six directions transverse to the giant, i.e. the X^-, radial and X^a directions. The mass of physical modes are all integer multiples of μ, a characteristic scale of the background, and are independent of the radius of the brane. In this section, we consider the (anti-)selfdual B-field case and compute the effects of the B-field on the spectrum of fluctuations of the squashed giant graviton in the X^i and X^a directions. In particular, as a confirmation of the spatial diffeomorphism invariance of the Born-Infeld action we show the existence of (three) zero modes. Furthermore, we find that the spectrum of the squashed giant, similarly to the unsquashed round case, is still independent of p^+ and g_s and is only a function of the B-field.

3.1 Corrections to the spectrum of X^i modes

As we saw in the section 2.3, in an (anti-)selfdual background B-field the brane retains a spherical shape, changing only its size. In this case, the shape parameter s introduced in the section 2.3, is identically zero, and the embedding of the brane (2.38) takes a form

$$X^i = r(B)\, R_0 x^i \,. \tag{3.1}$$

The potential (2.35) reduces to (2.53), and $r(B)$ in (3.1) at the minimum of the potential must satisfy the condition

$$1 - 4r_{\min}^2 + 3\left(1 + \frac{B^2}{4}\right) r_{\min}^4 = 0 \,, \tag{3.2}$$

which has the solution (2.55). Let's parametrize small fluctuations around the embedding (3.1) by Z^i, i.e., set

$$X^i = r(B)\, R_0 x^i + Z^i \,. \tag{3.3}$$

As shown in [9], all fluctuations of the shape can be expanded in terms of SO(4) spherical harmonics which are the eigenmodes of the operator $\mathcal{L}_{ij} = x_j\partial_i - x_i\partial_j$. We will assume that Z^i in (3.3) has an eigenvalue λ, explicitly

$$\mathcal{L}_{ij} Z^j = \lambda\, Z^i \,. \tag{3.4}$$

JHEP07(2004)077

Using the definition of $\mathcal{L}_{ij}$ and the properties (2.10), one can rewrite (3.4) in the form

$$\epsilon_{ijkl}\{x^j, x^k, Z^l\} = -2\lambda Z^i\,, \tag{3.5}$$

or, equivalently,

$$\{x^j, x^k, Z^l\} + \{x^k, x^l, Z^j\} + \{x^l, x^j, Z^k\} = -\lambda\epsilon_{ijkl}\, Z^i\,. \tag{3.6}$$

Plugging (3.3) into (2.11) and using the formulas (3.5) and (3.6), up to the second order in Z we get[8]

$$V = \frac{\mu^2 p^+}{2}\left[1 + 4\lambda r^2 + r^4\lambda(\lambda-2)\left(1+\frac{B^2}{4}\right)\right] Z^i Z^i\,, \tag{3.7}$$

where we have also used the fact that $T_{ij}(B) = 0$ for an (anti-)selfdual B-field. Now, using the extremum condition (3.2), we can rewrite (3.7) as

$$V = \frac{\mu^2 p^+}{6}(\lambda+1)\left[(4r^2_{\rm min}-1)\lambda + 3\right] Z^i Z^i\,. \tag{3.8}$$

From this formula, noting that the kinetic term of the light-cone hamiltonian is $P^2/2p^+$, one can read off the frequency of the eigenmodes:

$$\omega^2_\lambda = \mu^2(\lambda+1)\left[\frac{\lambda}{3}(4r^2_{\rm min}-1)+1\right]\,. \tag{3.9}$$

As discussed in [9], λ can take three values: $\lambda = -1$,

$$\lambda = l\,, \qquad l = 0, 1, 2, \dots\,, \tag{3.10}$$

and

$$\lambda = -(l+2)\,, \qquad l = 1, 2, \dots\,, \tag{3.11}$$

where l is the SO(4) quantum number. The modes with $\lambda = -1$ are zero modes and represent the non-physical gauge degrees of freedom.[9] There are two sets of physical modes corresponding to the values of λ (3.10) and (3.11). Noting that $0 \le B < B_{cr} = \sqrt{4/3}$ which leads to $\frac{1}{2} < r^2_{\rm min} \le 1$ or $1 < 4r^2_{\rm min} - 1 \le 3$, it is easy to see that all the non-zero modes for all values of l have positive ω^2, and therefore there are no tachyons in the spectrum. It is worth noting that the frequency of the $l = 0$ mode, which corresponds to the center of mass motion of the squashed giant without changing its shape, is independent of the B-field and squashing.

[8] We also use here the integration by parts in the form $\int A\{B, C, D\} = -\int B\{A, C, D\}$.

[9] The existence of three zero modes is very important and is a cross check for the correctness of our calculations. One can then repeat the above calculations for the generic B-field. Inserting the expansion

$$X^i = D_{ij}(R_0 x^j + Z^j)$$

into the potential, assuming that $\mathcal{L}_{ij} Z_j = -Z^i$ (the "zero-modes") and expanding to the second order in Z, we obtain

$$V^{(2)}|_{\rm zero\ mode} = \frac{\mu^2 p^+}{2} Z_i D_{ik}\frac{\partial V}{\partial D_{kj}} Z_j$$

where V is given in (2.34). It is then evident that for the squashed giant configuration, where $\partial V/\partial D_{ij}$ vanishes, Z_i's satisfying $\mathcal{L}_{ij} Z_j = -Z_i$ are zero modes.

For the $B = 0$ case ($r_{\min} = 1$) both of the modes with $\lambda = l$ and $\lambda = -(l+2)$ have the same mass $\mu(l+1)$ [9]. This degeneracy is, however, lifted due to the resizing corrections, and these two modes have now different masses.

For the critical B-field where $r^2_{\min} = 1/2$, the $l = 1$, $\lambda = -3$ mode becomes massless while all the other modes still have a positive mass squared. This is a sign that the squashed giant would become unstable for $B > B_{cr}$. Furthermore, this shows that the radial (breathing) mode is the mode along which the squashed giant develops instability.

Before moving to the other modes, we would like to point out that the above "small fluctuation" expansion may break down for the modes with high l. More precisely, the above expansion can be trusted for the modes for which

$$\omega_l \lesssim \mu \frac{B^2_{cr} - B^2}{g^2_{\text{eff}}} .$$

In terms of the SO(4) quantum number l, that is $l \lesssim \frac{3(B^2_{cr}-B^2)}{(4r^2_{\min}-1)g^2_{\text{eff}}}$.

3.2 Corrections to the spectrum of X^a modes

Analogously to the X^i modes, corrections to the spectrum of X^a modes can be computed. Consider small fluctuations around the solution $X^i = R_0 r x^i$, $X^a = 0$, i.e., plug $X^i = R_0 r x^i$, $X^a = Z^a$ into the potential (2.4), (2.7) and keep only terms of the second order in Z^a:

$$V = \frac{\mu^2 p^+}{2}\left[Z^a Z^a + \frac{r^4_{\min}}{2}\left(\{x^i, x^j, Z^a\}\{x^i, x^j, Z^a\} + \{x^i, x^k, Z^a\}\{x^j, x^l, Z^a\}B_{ij}B_{kl}\right)\right]. \tag{3.12}$$

To compute this expression, we go to the basis where an anti-selfdual B-field has the only non-zero components $B_{12} = -B_{34} = B/2$, and assume that Z^a are SO(4) harmonics with quantum numbers (l, m_1, m_2) defined as[10]

$$\mathcal{L}_{ij}\mathcal{L}_{ij} Z^a = -2l(l+2)\ Z^a\,, \tag{3.13}$$

$$\mathcal{L}_{12} Z^a = i m_1\, Z^a\,, \qquad \mathcal{L}_{34} Z^a = i m_2\ Z^a\,, \tag{3.14}$$

where $-l \le m_1, m_2 \le l$. Now, using the formulas

$$\{x^i, x^j, \Phi\} = -\frac{1}{2}\epsilon^{ijkl}\mathcal{L}_{kl}\Phi \tag{3.15a}$$

$$\epsilon^{ijkl}\mathcal{L}_{ij}\mathcal{L}_{kl}\Phi = 0\,, \tag{3.15b}$$

(the identity (3.15b) can be explicitly verified using definition of $\mathcal{L}_{ij}$) and integrating by parts, we get

$$V = \frac{\mu^2 p^+}{2}\ \left[1 + r^4_{\min}\left(l(l+2) + \frac{B^2}{4}(m_1 - m_2)^2\right)\right] Z^a Z^a\,. \tag{3.16}$$

[10]Note that these numbers must be the same for all components Z^a, since the rotations in the X^i and X^a directions commute with each other. Note also that with our definition $\mathcal{L}_{ij}$ is anti-hermitian and hence eigenvalues of $\mathcal{L}^2$ are negative and eigenvalues of $\mathcal{L}_{12}$ and $\mathcal{L}_{34}$ are imaginary.

Again, this expression is positive for all (l, m_1, m_2), and there are no tachyons. The frequencies (masses) are

$$\omega^2_{lm_1m_2} = \mu^2\left[1 + r^4_{\min}\left(l(l+2) + \frac{B^2}{4}(m_1 - m_2)^2\right)\right]. \tag{3.17}$$

This should be contrasted with the masses of the X^i fluctuations which have only l dependence. Note also that the azimuthal dependence of the frequencies only appears in the combination $m = m_1 - m_2$, and not m_1 and m_2 individually. This is related to the fact that in the self-dual B-field case we remain with a $\mathrm{SU}(2) \times \mathrm{U}(1)$ symmetry.

It is straightforward to compute the masses for the general $B_{12} \neq \pm B_{34}$ case. In this case, however, one should note that besides the resizing parameter r, the frequencies also depend on the reshaping parameter s. Performing the calculations we obtain

$$\begin{aligned}\omega^2_{lm_1m_2} = \mu^2\Big[&1 + l(l+2)(r^2 - s^2)^2 + 4rs\left((r+s)^2 m_2^2 - (r-s)^2 m_1^2\right) + \\ &+ \left((r+s)^2 B_{12} m_2 + (r-s)^2 B_{34} m_1\right)^2\Big].\end{aligned} \tag{3.18}$$

The above expression reproduces (3.17) for $s = 0$ and $B_{12} = -B_{34}$. It is interesting to note that in the limit (2.41) where the minimum is at $r = \pm s$, the spectrum (3.18) becomes l independent, moreover the spectrum would only depend on m_1 or m_2 (depending on the $+/-$ sign). This is compatible with the arguments at the end of section 2.3.

4. Stability of the squashed giant

In previous sections we studied deformation of the shape and corrections to the spectrum of a giant graviton due to the presence of a constant background NSNS B-field. In this section we address the stability of the squashed giant. As mentioned in previous section, there are no tachyonic modes in the spectrum of the geometric fluctuations of the squashed giant. This implies the classical stability of the squashed giant. Of course one should remember that even classically the giant is not stable under the fluctuations whose energy are comparable to $\frac{B^2_{cr} - B^2}{g^2_{\text{eff}}}$.

In section 4.1 we analyze the supersymmetry of the squashed giant, and in section 4.2 we study instability of the squashed giant under quantum (tunneling) effects.

4.1 SUSY analysis

To study the supersymmetry of the squashed giant, we first start with the supersymmetry algebra of the background plane-wave. The plane-wave (1.1) is a maximally supersymmetric background with 32 supercharges half of which are kinematical anti-commuting to the light-cone momentum p^+, and the other half are dynamical which anti-commute to the light-cone hamiltonian, explicitly

$$
\begin{aligned}
[P^+, Q] &= 0\,, \qquad [P^+, Q^\dagger] = 0 \\
[H, Q] &= 0\,, \qquad [H, Q^\dagger] = 0\,. \\
\{Q_{\alpha\dot\beta}, Q^{\dagger\rho\dot\lambda}\} &= \delta^\rho_\alpha \delta^{\dot\lambda}_{\dot\beta}\mathbf{H} + 2\mu(i\sigma^{ij})^\rho_\alpha \delta^{\dot\lambda}_{\dot\beta}\mathbf{J_{ij}} + 2\mu(i\sigma^{ab})^{\dot\lambda}_{\dot\beta}\delta^\rho_\alpha \mathbf{J_{ab}} \\
\{Q_{\dot\alpha\beta}, Q^{\dagger\dot\rho\lambda}\} &= \delta^{\dot\rho}_{\dot\alpha} \delta^{\lambda}_{\beta}\mathbf{H} + 2\mu(i\sigma^{ij})^{\dot\rho}_{\dot\alpha} \delta^{\lambda}_{\beta}\mathbf{J_{ij}} + 2\mu(i\sigma^{ab})^{\lambda}_{\beta}\delta^{\dot\rho}_{\dot\alpha} \mathbf{J_{ab}}
\end{aligned}
\tag{4.1}
$$

where Q's are dynamical supercharges and the two indices on them are Weyl index of the two SO(4)'s acting on the X^i and X^a directions. The above superalgebra can be identified as PSU(2|2) $\times$ PSU(2|2) $\times$ U(1)$_-$ $\times$ U(1)$_+$ where the U(1)$_\pm$ are translation along the $x^\pm$ directions, generators of which are $p^+, H_{l.c}$. More detailed discussion on this superalgebra can be found in [8]. Adding the B-field does not change the supergravity background and hence its superalgebra.

In the absence of the B-field a round three sphere giant is half supersymmetric, and it preserves all the dynamical supercharges. This can be readily seen from the fact that this solution is a zero energy solution ($H = 0$) and has zero charges under both of the SO(4)'s, i.e. it has $J_{ij}, J_{ab} = 0$. Therefore, acting on the round giant graviton state, the right-hand side of (4.1) vanishes.

Fluctuations of the round giant are generically less supersymmetric states. In particular let us focus on the two modes which appear in the reshaping and resizing. In the notations of [9] both of these have $l = 1$ under one of the SO(4)'s and have zero charge under the other SO(4). The resizing mode (the "breathing" mode), $\delta X^i = x^i$ has zero J_{ij} and J_{ab} charge with energy $H = 2\mu$. Hence this mode by itself is not BPS at all (acting by this state, the right-hand-side of (4.1) is non-vanishing). The reshaping mode $\delta X^i = S_{ij}x^j$ which comes with degeneracy $3\times 3 = 9$ also has energy 2μ and $J_{ab} = 0$ whereas its J_{ij} eigenvalues can be 0 or ± 1. Therefore among the nine modes of the reshaping mode there are two modes which kill the right-hand-side of (4.1) for half of the dynamical supercharges, i.e. they are 1/4 BPS. (For a similar argument for SU(4|2) superalgebra see [19].) It is straightforward to check that the reshaping and resizing modes fall into the same supermultiplet of the PSU(2|2) $\times$ PSU(2|2) $\times$ U(1)$_-$ superalgebra which contains a 1/4 BPS state [9]. In other words this multiplet is a 1/4 BPS (short) multiplet.

Now let us consider the squashed giant. Although the state of squashed giant does not kill the right-hand-side of the SUSY algebra (4.1), as we argued above, the deformation of the giant from a round three sphere can be described in terms of turning on a 1/4 BPS multiplet. In this sense the squashed giant is 1/4 BPS. This should be contrasted with the flat space case, where a D-brane in the background B-field preserves the same amount of supersymmetry as a usual D-brane, i.e. half BPS. Being 1/4 BPS one would expect that the shape of the squashed giant, at least perturbatively, should be stable (protected) under small geometric fluctuations of the giant. However, there is a subtlety which despite of being BPS might make the shape unstable: in principle it is possible that some short multiplets combine and form a long (non-BPS) multiplet and hence receive corrections. Based on similar analysis which was done for a spherical membrane in the eleven dimensional plane-wave background [19], however, we expect the specific modes involved in the reshaping of the giant to be stable. Indeed this expectation is well supported noting the potential (2.34) and the fact that the $g_{\rm eff}$ dependence of the potential is only in an overall factor. Therefore the value of (r,s) which minimize the potential, and hence the shape and size of the giant, is independent of $g_{\rm eff}$. In other words, the shape of the brane is stable under perturbative corrections about $g_{\rm eff}=0$. This, however, does not exclude non-perturbative instabilities of the giant.

4.2 Quantum instability of the squashed giant

Supersymmetry considerations imply that the shape of the squashed giant should be perturbatively stable. However, as one can explicitly see from (2.35) and the potentials depicted in section 2, the energy of the squashed giant is non-zero cf. (2.27) while the minimum at $X=0$ has always zero energy. This in particular means that the squashed giant can tunnel to the X=0 vacuum. To have an estimate of the tunneling rate let us focus on the (anti-)selfdual B-field case and the potential (2.53), depicted in figure 6.

Using the WKB approximation the tunneling probability is equal to the negative of the exponential of the area encapsulated between the potential V_{ASD} and line $r=r_{\rm min}$:

$$P_{\rm tunnel}\approx e^{-\frac{B_{cr}^2-B^2}{g_{\rm eff}^2}\Delta r(B)} \tag{4.2}$$

where $\Delta r(B)=r_0-r_{\rm min}$ with $V(r_0)=V_{\rm min}$ and $\Delta r(B_{cr})=0$. The dependence of tunneling probability on $g_{\rm eff}$ is like $e^{-1/g_{\rm eff}^2}$ and hence from the giant graviton (gauge theory) viewpoint this tunneling is an instanton effect. Therefore the squashed giant is metastable and through instanton (non-perturbative) effects would tunnel into the stable $X=0$ vacuum where it decays into the supergravity modes.

5. Discussion

In this paper we discussed giant gravitons in the ten dimensional type-IIB plane-wave background when a constant NSNS B-field is turned on along the giant. As we argued if the B-field has only one of its legs along the three-sphere giant it is not essentially felt by the giant (similarly to the flat D-brane case [12].) Moreover, the shape of the giant would change as a result of the existence of the B-field so that generically we have a "squashed giant".

As we showed in section 2, there is a critical B-field, B_{cr}, above which the squashed giant becomes classically unstable. This means that if we tune up the B-field from zero we start squashing (resizing and reshaping) the giant and at $B>B_{cr}$ the squashed giant configuration is no longer minimizing the potential. Hence in the $B>B_{cr}$ background the only minimum is at $X^i=X^a=0$ (the zero size brane). As argued in [1, 9] this $X=0$ solution cannot be stable as a quantum mechanical vacuum of the theory and it is not clear yet what this vacuum would look like quantum mechanically.

As we discussed and can be seen in figures 1, 2 and 6 the potential has a maximum and one would wonder whether it is possible to expand the theory around the maximum of the potential where we have a system with open string tachyons. The giant graviton state is then where the tachyon is condensed (of course there is also the possibility that the tachyon rolls towards the $X = 0$ vacuum). Although the above argument is generic for $B = 0$ or $B \neq 0$, the non-zero B-field case has its own special and interesting features. Let us focus on the selfdual B-field case. As it is seen from figure 6, it is possible to consider the interesting limit of $B \to B_{cr}$ ($B < B_{cr}$) and $g_{\rm eff} \to 0$ while $(B_{cr}^2 - B^2)/g_{\rm eff}^2$ is kept fixed. In this limit the difference between the energies of the squashed giant and the $X = 0$ vacuum is sent to infinity while keeping the energy difference between the squashed giant minimum and the maximum of the potential finite. In this limit almost all the modes of the geometric fluctuations of the giant are also decoupled (note that $g_{\rm eff} \to 0$) and we remain with the $l = 1$ modes. The potential in figure 6 in this limit resembles that of a $c = 1$ matrix model [20]. It is then very plausible to expect that the squashed giant system would provide us with a laboratory to study open string tachyon condensation.

Here we mainly focused on the squashed giants in the plane-wave background, one might pose the same problem in the $AdS_5 \times S^5$ background, for which we again expect to see a similar squashing behaviour.

The other interesting open question is: what is the $\mathcal{N} = 4$ gauge theory operator which is dual to the squashed giant? A simple proposal, based on the results we obtained here and the discussions of [5, 9], is that this operator is a subdeterminant type operator [5] in the BMN sector [8, 21] which has the appropriate $\mathrm{SO}(4) \times \mathrm{SO}(4)$ charges. More specifically, this operator should be obtained by the insertion of the covariant derivative of the gauge theory on $R \times S^3$, $\mathcal{D}_i$, in the combinations $T_{ij} = F_{ij}^2 - \delta_{ij} F^2/4$ where $F_{ij} = [\mathcal{D}_i, \mathcal{D}_j]$ (this part would give the reshaping) and F^2 terms (for resizing), into the subdeterminant operators. Working out the explicit form of this operator is an interesting open question we postpone to future works. Once we have this operator one might then compute the decay rate of the squashed giant, an approximation of which is given in (4.2), from the dual gauge theory viewpoint.

Another interesting question which we briefly discussed is quantization of the squashed giant and also the (noncommutative) gauge theory living on the giant. This would generically lead to a quantization of the background B-field (cf. (2.62)). This direction deserves a more thorough and detailed analysis.

A problem similar to what we considered here for three-sphere giants may be asked about the spherical M5-branes in the eleven dimensional plane-wave [19, 21] or $AdS_{4,7} \times S^{7,4}$ background [1]. In analogy with our results for the three-sphere case, we expect in response to a background constant three form field the five sphere giant to be deformed (squashed), moreover we expect this "squashed" five sphere giant to be a bound state of spherical M2-brane and spherical M5-brane giants. The theory residing on the "squashed" five sphere giant is then a deformation of the $(0, 2)$ theory on $R \times S^5$. It would be nice to check if this deformation would provide us with an expansion parameter which could be used to make a perturbative analysis of the $(0, 2)$ theory.

Acknowledgments

We would like to thank Simeon Hellerman, Keshav Dasgupta, John McGreevy, Darius Sadri and Matt Strassler for fruitful discussions and Michal Fabinger for his collaboration at the early stages of this work. The work of M.M.Sh-J. is supported in part by NSF grant PHY-9870115 and in part by funds from the Stanford Institute for Theoretical Physics. The work of S.P. is supported by NSF grant PHY-0244725.

A. Some useful identities

The B-field, B_{ij} $i,j=1,2,3,4$, we have been working with is in $\mathbf{6}$ of SO(4). In terms of SU(2)$\times$SU(2) representations, however, this is a reducible one. That is, we can decompose B into its self-dual and anti-self-dual parts, B^+ and B^- respectively:

$$B^+_{ij}=\frac{1}{2}\left(B_{ij}+\frac{1}{2}\epsilon_{ijkl}B_{kl}\right), \qquad B^-_{ij}=\frac{1}{2}\left(B_{ij}-\frac{1}{2}\epsilon_{ijkl}B_{kl}\right). \tag{A.1}$$

B^+ and B^- are then in $(\mathbf{3},\mathbf{1})$ and $(\mathbf{1},\mathbf{3})$ of SU(2) $\times$ SU(2).

The "energy-momentum" tensor T_{ij},

$$T_{ij}=B_{ik}B_{kj}+\frac{1}{4}\delta_{ij}B^2\,,$$

which is in $(\mathbf{3},\mathbf{3})$ of SU(2) $\times$ SU(2), takes a simple form once $B^\pm$ are used:

$$T_{ij}=2B^+_{ik}B^-_{kj}=2B^-_{ik}B^+_{kj}\,. \tag{A.2}$$

Using (A.2) and the fact that $B^\pm_{ik}B^\pm_{jk}=\frac{1}{4}B^2\delta_{ij}$ we obtain a very useful identity

$$T_{ik}T_{jk}=\frac{1}{4}T^2\delta_{ij}\,, \tag{A.3}$$

where

$$T^2\equiv T_{ij}T_{ij}=(B^+)^2(B^-)^2\,. \tag{A.4}$$

Noting that

$$B^2=(B^+)^2+(B^-)^2 \tag{A.5}$$

and the above, $(B^+)^2$ and $(B^-)^2$ are then solutions of the quadratic equation $X^2-B^2X+T^2=0$.

Let us define the matrix D_{ij},

$$D_{ij}=r\delta_{ij}+\lambda T_{ij}\,, \tag{A.6}$$

where r and λ are two arbitrary c-numbers. It is straightforward to show that

$$\begin{aligned}
\operatorname{Tr}D&=4r\,,\qquad \operatorname{Tr}D^2=4r^2+\lambda^2T^2\,,\\
\operatorname{Tr}D^3&=3r\operatorname{Tr}D^2-8r^3=4r^3+3r\lambda^2T^2\,,\\
\operatorname{Tr}D^4&=\frac{1}{4}\left(\operatorname{Tr}D^2\right)^2+4r^2\operatorname{Tr}D^2-16r^4\,,\\
\operatorname{Tr}D^6&=\operatorname{Tr}D^2\left[\frac{1}{16}\left(\operatorname{Tr}D^2\right)^2+3r^2\operatorname{Tr}D^2-12r^4\right],\\
\det D&=\frac{1}{16}(\operatorname{Tr}D^2-8r^2)^2=\frac{1}{16}(4r^2-T^2\lambda^2)^2\,,
\end{aligned} \tag{A.7}$$

$$B_{ik}T_{kj}B_{ji}=T^2\,,\qquad T_{ik}B_{kj}T_{jl}B_{li}=-\frac{1}{4}B^2T^2\,, \tag{A.8}$$

and

$$
\begin{aligned}
\operatorname{Tr}(D^n B) &= 0\,, \qquad n = 0,1,2,\dots\,, \\
\operatorname{Tr}(D^2BD^2B) &= -B^2\left[\frac{1}{16}\left(\operatorname{Tr}D^2\right)^2 + r^2\operatorname{Tr}D^2 - 4r^4\right] + r\lambda T^2\operatorname{Tr}D^2\,, \\
\operatorname{Tr}(D^4BD^2B) &= -B^2\operatorname{Tr}D^2\left[\frac{1}{64}\left(\operatorname{Tr}D^2\right)^2 + \frac{3}{4}r^2\operatorname{Tr}D^2 - 3r^4\right] + \\
&\quad + r\lambda T^2\left[\frac{3}{8}\left(\operatorname{Tr}D^2\right)^2 + 2r^2\operatorname{Tr}D^2 - 8r^4\right]. \qquad \text{(A.9)}
\end{aligned}
$$

B. Squashed three sphere

A round three-sphere has $SO(4) \simeq SU(2)_L \times SU(2)_R$ symmetry (isometries), however, only a part of this symmetry group can be made explicit in the metric once a coordinate system is adopted. For example in the coordinate system in which the line element is $R^2(d\theta^2 + \sin^2\theta d\Omega_2^2)$, $SU(2)_D$, i.e. the diagonal part of $SU(2)_L$ and $SU(2)_R$ is manifest. It is possible to adopt another coordinate system in which $SU(2)_L \times U(1)$ is explicit:

$$
\begin{aligned}
z_1 &= R\cos\frac{\theta}{2}\,e^{i(\phi+\psi)/2} \\
z_2 &= R\sin\frac{\theta}{2}\,e^{i(-\phi+\psi)/2}
\end{aligned} \qquad \text{(B.1)}
$$

with $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$ and $0 \le \psi \le 4\pi$, which gives the embedding of a S^3 in $\mathbb{C}^2$. $(z_1\, z_2)$ behave like a doublet under $SU(2)_L$, and $U(1)$ is the translation along ψ direction. In this coordinate system the line element on S^3 is

$$
d\Omega_3^2 = \frac{R^2}{4}\left[(d\theta^2 + \sin^2\theta d\phi^2) + (d\psi + \cos\theta d\phi)^2\right]. \qquad \text{(B.2)}
$$

In this coordinate system S^3 is realized as $U(1)$ fiber over an S^2 base. The S^2 base and S^1 fiber have the same radii equal to $R/2$.

Squashed three sphere, S_q^3 is a deformation of round S^3 with $SU(2) \times U(1)$ isometry. The metric for the squashed three sphere can be written as

$$
d\Omega_q^2 = \frac{R_q^2}{4}\left[(d\theta^2 + \sin^2\theta d\phi^2) + q^2(d\psi + \cos\theta d\phi)^2\right], \qquad \text{(B.3)}
$$

where q is the deformation (squashing) parameter and may be taken smaller or larger than one.

C. Effective closed string coupling in the plane-waves

It is well-known that upon compactification of strings the effective coupling, i.e. effective Newton constant, for strings in lower dimensions and the original uncompactified theories are related by a factor of $\sqrt{V}$, where V is the volume of the compactification manifold, in such a way that for infinite V case the coupling of the lower dimensional theory is vanishing (note that this is only true when there is no warping factor in the non-compact part).

JHEP07(2004)077

For the case of strings on the plane-wave, however, because of the "harmonic oscillator potential" (arising from the $(dx^+)^2$ term in metric once the light-cone gauge is employed) in a sense the situation is very similar to a compactification on an eight dimensional manifold [22]. To see this more concretely, let us consider a scalar field theory on the ten dimensional plane-wave [23]. The classical equations of the scalar field, once the interaction terms are turned off, can be exactly solved and the dependence of the wavefunction on the transverse coordinates is given in terms of harmonic oscillator wavefunctions, or Hermit polynomials while it is like a free particle moving in the x^+ and x^- directions [23]. Therefore, the particles in the plane-wave background can only freely move along $x^\pm$ directions and (depending on their light-cone momentum p^+) they have only access to a finite volume in the transverse directions. (One may then consider interactions, and analyze the theory effectively as a two dimensional non-local, non-relativistic field theory which we do not intend to do here. For further discussions on this theory see [23]).

Now, let us consider ten dimensional type-IIB supergravity on the plane-wave background. In order to read off the effective coupling of strings, or the effective Newton constant (of course as explained above in the two dimensional sense) one should work out the "effective" compactification volume in the plane-wave background. As discussed in [23] one should insert a factor of $(1/\sqrt{\mu p^+})^d$, where d is the number of transverse directions, for *the* plane-wave background $d = 8$. From this, noting that in the string frame (and in string units) $G_N^{(10)} = g_s^2$, the two dimensional effective Newton constant, $G_N^{(2)}$ is [22]

$$G_N^{(2)} = g_s^2(\mu p^+)^4 \,. \tag{C.1}$$

In the plane-wave analysis, however, motivated by the dual gauge theory computations, it has been more customary to use g_2 instead, where $g_2^2 \equiv G_N^{(2)}$ and hence

$$g_2 = g_s(\mu p^+)^2 \,. \tag{C.2}$$

In terms of the dual gauge theory parameters [8],

$$(\mu p^+)^2 g_s = \frac{J^2}{N} \,, \tag{C.3}$$

and hence $g_2 = J^2/N$. Performing direct dual gauge theory calculations it has been confirmed that g_2 is indeed the genus counting parameter in the BMN sector (i.e. the sector of $\mathcal{N} = 4$ U(N) SYM gauge theory which consists of operators carrying J units of R-charge).

References

[1] J. McGreevy, L. Susskind and N. Toumbas, *Invasion of the giant gravitons from anti-de Sitter space*, *J. High Energy Phys.* **06** (2000) 008 [hep-th/0003075].

[2] M.T. Grisaru, R.C. Myers and O. Tafjord, *SUSY and goliath*, *J. High Energy Phys.* **08** (2000) 040 [hep-th/0008015].

[3] A. Hashimoto, S. Hirano and N. Itzhaki, *Large branes in AdS and their field theory dual*, *J. High Energy Phys.* **08** (2000) 051 [hep-th/0008016].

[4] S.R. Das, A. Jevicki and S.D. Mathur, *Vibration modes of giant gravitons*, *Phys. Rev.* **D 63** (2001) 024013 [hep-th/0009019];
R.C. Myers and O. Tafjord, *Superstars and giant gravitons*, *J. High Energy Phys.* **11** (2001) 009 [hep-th/0109127];
F. Leblond, R.C. Myers and D.C. Page, *Superstars and giant gravitons in M-theory*, *J. High Energy Phys.* **01** (2002) 026 [hep-th/0111178];
P. Ouyang, *Semiclassical quantization of giant gravitons*, hep-th/0212228;
B. Janssen, Y. Lozano and D. Rodriguez-Gomez, *A microscopical description of giant gravitons, II. The $AdS_5 \times S^5$ background*, *Nucl. Phys.* **B 669** (2003) 363 [hep-th/0303183].

[5] V. Balasubramanian, M. Berkooz, A. Naqvi and M.J. Strassler, *Giant gravitons in conformal field theory*, *J. High Energy Phys.* **04** (2002) 034 [hep-th/0107119];
S. Corley, A. Jevicki and S. Ramgoolam, *Exact correlators of giant gravitons from dual $N = 4$ SYM theory*, *Adv. Theor. Math. Phys.* **5** (2002) 809 [hep-th/0111222];
D. Berenstein, *Shape and holography: studies of dual operators to giant gravitons*, *Nucl. Phys.* **B 675** (2003) 179 [hep-th/0306090].

[6] V. Balasubramanian, M.-x. Huang, T.S. Levi and A. Naqvi, *Open strings from $N = 4$ super Yang-Mills*, *J. High Energy Phys.* **08** (2002) 037 [hep-th/0204196].

[7] J.C. Plefka, *Lectures on the plane-wave string/gauge theory duality*, *Fortschr. Phys.* **52** (2004) 264 [hep-th/0307101];
J.M. Maldacena, *Tasi 2003 lectures on AdS/CFT*, hep-th/0309246;
M. Spradlin and A. Volovich, *Light-cone string field theory in a plane wave*, hep-th/0310033.

[8] D. Sadri and M.M. Sheikh-Jabbari, *The plane-wave/super Yang-Mills duality*, hep-th/0310119.

[9] D. Sadri and M.M. Sheikh-Jabbari, *Giant hedge-hogs: spikes on giant gravitons*, *Nucl. Phys.* **B 687** (2004) 161 [hep-th/0312155].

[10] R.R. Metsaev and A.A. Tseytlin, *Exactly solvable model of superstring in plane wave Ramond-Ramond background*, *Phys. Rev.* **D 65** (2002) 126004 [hep-th/0202109].

[11] E. Gava, K.S. Narain and M.H. Sarmadi, *On the bound states of p- and (p+2)-branes*, *Nucl. Phys.* **B 504** (1997) 214 [hep-th/9704006];
M.M. Sheikh-Jabbari, *More on mixed boundary conditions and D-branes bound states*, *Phys. Lett.* **B 425** (1998) 48 [hep-th/9712199].

[12] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[13] O. Lunin, S.D. Mathur, I.Y. Park and A. Saxena, *Tachyon condensation and 'bounce' in the D1-D5 system*, *Nucl. Phys.* **B 679** (2004) 299 [hep-th/0304007];
J.M. Camino and A.V. Ramallo, *Giant gravitons with NS-NS B-field*, *J. High Energy Phys.* **09** (2001) 012 [hep-th/0107142].

[14] A. Mikhailov, *Giant gravitons from holomorphic surfaces*, *J. High Energy Phys.* **11** (2000) 027 [hep-th/0010206];
Nonspherical giant gravitons and matrix theory, hep-th/0208077;
D. Bak, *Supersymmetric branes in pp wave background*, *Phys. Rev.* **D 67** (2003) 045017 [hep-th/0204033].

JHEP07(2004)077

[15] F. Ardalan, H. Arfaei and M.M. Sheikh-Jabbari, *Noncommutative geometry from strings and branes*, *J. High Energy Phys.* **02** (1999) 016 [hep-th/9810072];
C.-S. Chu and P.-M. Ho, *Noncommutative open string and D-brane*, *Nucl. Phys.* **B 550** (1999) 151 [hep-th/9812219].

[16] M.R. Douglas and C.M. Hull, *D-branes and the noncommutative torus*, *J. High Energy Phys.* **02** (1998) 008 [hep-th/9711165];
M.M. Sheikh-Jabbari, *Super Yang-Mills theory on noncommutative torus from open strings interactions*, *Phys. Lett.* **B 450** (1999) 119 [hep-th/9810179].

[17] M.M. Sheikh-Jabbari, *A note on the deformation of lambda-symmetry in B-field background*, *Phys. Lett.* **B 477** (2000) 325 [hep-th/9910258];
N. Seiberg, *A note on background independence in noncommutative gauge theories, matrix model and tachyon condensation*, *J. High Energy Phys.* **09** (2000) 003 [hep-th/0008013].

[18] B. Ydri, *Fuzzy physics*, hep-th/0110006.

[19] K. Dasgupta, M.M. Sheikh-Jabbari and M. Van Raamsdonk, *Protected multiplets of M-theory on a plane wave*, *J. High Energy Phys.* **09** (2002) 021 [hep-th/0207050].

[20] J. McGreevy and H. Verlinde, *Strings from tachyons: the $c = 1$ matrix reloaded*, *J. High Energy Phys.* **12** (2003) 054 [hep-th/0304224].

[21] D. Berenstein, J.M. Maldacena and H. Nastase, *Strings in flat space and pp waves from $N = 4$ super Yang-Mills*, *J. High Energy Phys.* **04** (2002) 013 [hep-th/0202021].

[22] N.R. Constable, D. Z. Freedman, M. Headrick, S. Minwalla, L. Motl, A. Postnikov and W. Skiba, *pp-wave string interactions from perturbative Yang-Mills theory*, *J. High Energy Phys.* **07** (2002) 017 [hep-th/0205089];
C. Kristjansen, J. Plefka, G.W. Semenoff and M. Staudacher, *A new double-scaling limit of $N = 4$ super Yang-Mills theory and pp-wave strings*, *Nucl. Phys.* **B 643** (2002) 3 [hep-th/0205033].

[23] D. Bak and M.M. Sheikh-Jabbari, *Strong evidence in favor of the existence of s-matrix for strings in plane waves*, *J. High Energy Phys.* **02** (2003) 019 [hep-th/0211073].